III Obstruction Theory and Classifying Bundles
(and Characteristic classes)

A. Obstruction Theory

we want to study sections of a fiber bundle

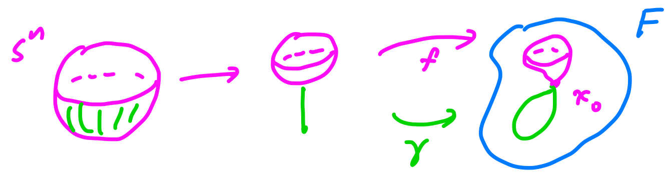
$$\begin{array}{ccc} F & \to & E \\ & & \downarrow p \\ & & M \end{array}$$

we assume: A) M is a CW complex (always true for manifolds)

B) F is n-simple for all n

i.e. action of $\pi_1(F, x_0)$ on $\pi_n(F, x_0)$ is trivial

recall, from Th^{m} I.13 action is



$[\gamma] \in \pi_1(F)$ $[f] \in \pi_n(F)$

exercise:

1) H-spaces (so Lie groups, loop spaces..) are n-simple
2) $\pi_1(F, x_0)$ is abelian if 1-simple
3) $\pi_n(F, x_0) \cong [S^n, F]$ homotopy classes of unbased maps

Th^{m} I.13

and a not so important assumption

C) action of $\pi_1(M, p(x_0))$ on $\pi_n(F, x_0)$ is trivial

$$\begin{tikzcd} & E \arrow[d, "p"] \\ {[0,1]} \arrow[r, "\gamma"] \arrow[ur, dashed] & M \end{tikzcd}$$ homotopy lifting

gives a map $\tilde{\gamma} : F \times [0,1] \to E$

so $\tilde{\gamma}(x,1) : \underset{F}{\underset{\shortparallel}{p^{-1}(x)}} \to \underset{F}{\underset{\shortparallel}{p^{-1}(x)}}$

so gives homo. $\tilde{\gamma}(x,1)_* : \pi_n(F) \circlearrowleft$

<u>exercise</u>: this implies $\pi_n(p^{-1}(x))$ is a fixed group independent of $x \in M$

(we can get around this by using "cohomology with local coefficients")

Denote the n-skeleton of M by $M^{(n)}$

assume we have a section $s_k : M^{(k)} \to E$

we define a cohomology class

$$\tilde{\sigma}(s_k) \in C^{k+1}(M); \pi_k(F))$$ CW-cochains

as follows

recall $\tau \in C^{k+1}(M; \pi_k(F))$ is a homomorphism

$$\tau : C_{k+1}(M) \to \pi_k(F)$$

and $C_{k+1}(M)$ is freely generated by the $(k+1)$-cells of M: $e_1^{k+1} \dots e_\ell^{k+1}$

recall, $M^{(k+1)} = M^{(k)} \cup e_1^{k+1} \cup \dots \cup e_\ell^{k+1} /\sim$

where $e_i^{k+1} = D^{k+1}$ and $a_i : \partial e_i^{k+1} \to M^{(k)}$

now $\sim$ above is gluing e_i^{k+1} to $M^{(k)}$ using a_i

and we have an "inclusion"

$$I_i : e_i^{k+1} \to M$$

since e_i^{k+1} is contractible, Cor II.3 says

$$\begin{tikzcd} I_i^* E \cong D^{k+1} \times F \arrow[d] \arrow[d] \\ e_i^{k+1} = D^{k+1} \end{tikzcd}$$

s_k pulls back to a section of $I_i^* E$ along ∂e_i^{k+1}

let $p_2 : I_i^* E \to F$ be projection

then $p_2 \circ s_k : \underset{\partial e_i^{k+1}}{\overset{\shortparallel}{S^k}} \to F$ gives an element of $\pi_k(F)$ (here we use B))

note: we also use C) so that all homotopy groups of all fibers are canonically identified

define $\tilde{\sigma}(s_k)(e_i^{k+1}) = [p_2 \circ s_k]$

so $\tilde{\sigma}(s_k) \in C^{k+1}(M, \pi_k(F))$

Remark: alternate def^n $I_i : \partial e_i^{k+1} \to M$ is null-homotopic (via I_i on D^{k+1}) and $s_k|_{\partial e_i^{k+1}}$ is a lift of the start of the homotopy. Homotopy lifting gives us a map $\tilde{H} : S^k \times [0,1] \to E$ and $\mathrm{im}(\tilde{H}(\cdot,1)) \subset p^{-1}(\underset{\text{constant}}{H(x,1)})$

exercise:

1) $\tilde{\sigma}(s_k)$ is invariant under homotopies of s_k

2) $\tilde{\sigma}(s_k) = 0 \iff s_k$ extends over $M^{(k+1)}$

lemma 1: $\boxed{\delta \tilde{\sigma}(s_k) = 0}$

for the proof recall in CW-homology

$$\partial^{cw} : C_{k+1}(M) \to C_k(M)$$

is defined as follows.

for each k-cell e_i^{k+1} consider

project to j^{th} k-cell

attaching map

$$\begin{tikzcd} \partial e_i^{k+1} = S^k \arrow[r, "a_i"] \arrow[rrr, bend right, "g_{ij}"'] & M^{(k)} \arrow[r] & M^{(k)}/M^{(k-1)} = \bigvee S^k \arrow[r] & S^k \end{tikzcd}$$

quotient map

one for each k-cell

\# k cells

$$\partial^{cw}(e_i^{k+1}) = \sum_{j=1}^{m} (\deg g_{ij}) [e_j^k]$$

exercise: we can homotop a_i so that there are open disks $D_n^k \subset S^k = \partial e_i^{k+1}$ so that

D_n^k maps homeomorphically onto interior of some k-cell and $a_i(\pm \partial D_n^k)$ is the attaching map of that k-cell, depending on D_n^k or^n pres or reversing

and $a_i(S^k - \bigcup D_n^k) \subset M^{(k-1)}$

Hint: homotop a_i so smooth on $a_i^{-1}(\mathrm{int}(k\text{-cells}))$ now take regular value in center of each k-cell (see proof of cellular approximation)

proof: recall $\delta(\tilde{\sigma}(s_k)) : C_{k+2}(M) \to \pi_k(F)$

$$e^{k+2} \longmapsto \tilde{\sigma}(s_k)(\partial e^{k+2})$$

let $a : \partial e^{k+2} \to M^{(k+1)}$ be the attaching map and

$I : e^{k+2} \to M$ the inclusion

homotop a as in exercise above

now $\delta \tilde{\sigma}(s_k)(e^{k+2}) = \tilde{\sigma}(s_k)(\sum d_j [e_j^{k+1}])$

as above
$$\begin{array}{c} I^* E \cong e^{k+2} \times F \\ \searrow \swarrow \\ e^{k+2} \end{array}$$

from exercise

and s_k gives a section above $a(\partial e^{k+2} - \cup D_n^{k+1})$

and $a|_{\partial D_n^{k+1}}$ is the attaching map for a

$(k+1)$-cell, say e_i^{k+1}

so can use $p_2 \circ a|_{\pm \partial D_n^{k+2}}$ to define $\tilde{\sigma}(s_k)(e_i^{k+1})$

$\therefore$ the maps used in the definition of

$$\sum d_j \tilde{\sigma}(s_k)(e_j^{k+1})$$

can be extended over $\partial e^{k+2} - \cup D_n^{k+1}$

exercise: show this means $\sum d_j \tilde{\sigma}(s_k)(e_j^{k+1})$

is 0 in $\pi_k(F)$

Hint: first case is $f: \partial D^{k+1} \to F$ is null-homotopic $\Leftrightarrow$ f extends over D^{k+1}

so $\delta \tilde{\sigma}(s_k) = 0$

now suppose we have 2 sections s_k and s_k' over $M^{(k)}$ that agree on $M^{(k-1)}$

then we define a difference class in $C^k(M, \pi_n(F))$

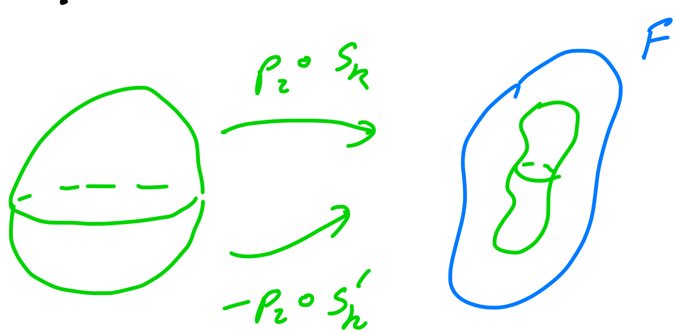
$$D(s_k, s_k') : C_k(M) \longrightarrow \pi_k(F)$$

as follows let $I_i : e_i^k \to M$ be the inclusion map

$$\begin{tikzcd}[column sep=small]
I_i^* E \arrow[dr] & \cong & e_i^k \times F \arrow[dl] \\
& e_i^k &
\end{tikzcd}$$

now $s_k|_{\partial e_i^k} = s_k'|_{\partial e_i^k}$ so we get

$$p_2 \circ \text{``} s_k|_{e_i^k} - s_k'|_{e_i^k} \text{''} : S^n \longrightarrow F$$



define $D(s_k, s_k')([e_i^k]) = [p_2 \circ s_k|_\partial - s_k'|_\partial] \in \pi_n(F)$

lemma 2:

1) $\delta(D(s_k, s_k')) = \tilde{\sigma}(s_k) - \tilde{\sigma}(s_k')$

2) given s_k and any $h \in C^k(M; \pi_k(F))$
$\exists\, s_k'$ st. $D(s_k, s_k') = h$

Proof: 1) is similar to proof of lemma 1

exercise: prove 1)

2) let s_k be any section on $M^{(k)}$

for a fixed k-cell e^k let

$$h: C_k(M) \to \pi_k(F)$$

be $h(e^k) = [g] \in \pi_k(F)$

and h of other k-cells be 0

(as e^k ranges over all cells such h will generate $C_k(M)$ so if we prove the result for this h we will be done)

let $I: e^k \to M$ be "inclusion"

as above
$$\begin{array}{ccc} I^*E & = & e^k \times F \\ & \searrow \swarrow & \\ & e^k & \end{array}$$

now choose a disk $D^k \subset \operatorname{int} e^k$ and homotop s_k on e^k so $p \circ s^k(D^k) = x_0 \in F$

let $s_k' = s_k$ on $M^{(k)} - D^k$ and on D^k

let it represent $-g \in \pi_k(F)$

(well $p_2 \circ s_k'$ represents $-g$)

clearly $D(s_k, s_k') = h$ ▦

the lemmas and discussion above prove

Th$^{\underline{m}}$ 3:

given a bundle $\begin{array}{c} F \to E \\ \downarrow p \\ M \end{array}$ satisfying A)–C) above

and a section $s_k: M^{(k)} \to E$, then

$$s_k|_{M^{(k-1)}} \text{ extends to } M^{(k+1)}$$

$$\Longleftrightarrow$$

$$\sigma(s_k) = [\tilde{\sigma}(s_k)] = 0 \in H^{k+1}(M; \pi_k(F))$$

So we have an obstruction to extending sections!

remark: if $\pi_k(F) = 0$ for $k < \dim M$ then the a above shows $\exists$ a section of $\begin{array}{c} F \to E \\ \downarrow \\ M \end{array}$

in particular, as noted above a bundle with contractible fibers always has a section!

note: $\sigma(s_k)$ depends on $s_k|_{M^{(k-1)}}$ i.e. it is not just about whether there is a section of E over M^{k+1} but whether our choice of section of $M^{(k)}$ (when restricted to $M^{(k-1)}$) extends to $M^{(k+1)}$

but the "first obstruction" is independent of any choices and is "natural"

Th^m 4:

given $\begin{array}{c} F \to E \\ \downarrow \\ M \end{array}$ satisfying A) - c) above

if $\pi_k(F) = 0$ for $k < n$, then $\exists$ a section $s_n : M^{(n)} \to E$ and the obstruction $\sigma(s_n)$ does not depend on s_n
(well-defined independent of choices)
denote $\sigma(s_n)$ by $\gamma^{n+1}(E)$
(called primary obstruction)

and if $f : N \to M$ is a map, then

$$\gamma^{n+1}(f^* E) = f^*(\gamma^{n+1}(E))$$

natural w.r.t. pull-back

γ^{n+1} is called a characteristic class

Proof: the discussion above says s_n exists (since all obstructions vanish)

you can develope an obstruction theory to homotoping one section to another
given s and s' with $s|_{M^{(k-1)}} = s'|_{M^{(k-1)}}$
then $s|_{M^{(k)}}$ is homotopic to $s'|_{M^{(k)}}$

$\Longleftrightarrow$

$\sigma(s, s') \in H^k(M; \pi_k(F))$ vanishes

so there is a unique section of E over $M^{(n-1)}$

$\therefore$ $\sigma(s_n)$ is well-defined

for naturallity suppose $f: N \to M$ is a cellular map (we can homotop to make it so)

now a section s of $E \to M$ gives a section $f^*(s)$ of $f^*(E)$ exercise: check this

for any $\Phi: (D^{n+1}, \partial D^{n+1}) \to (M^{(n+1)}, M^{(n)})$ we see

$$\pi_{n+1}(M^{(n+1)}, M^{(n)}) \longrightarrow \pi_{n+1}(N^{(n+1)}, N^{(n)}) \longrightarrow \pi_n(F)$$

$$[\Phi] \xmapsto{f_*} [f \circ \Phi] \longmapsto [p_2 \circ s \circ f \circ \Phi|_{\partial D^{n+1}}]$$

is essentially both $\sigma(f^* s)(\Phi)$ and $(f^* \sigma(s))(\Phi)$

now $\pi_{n+1}(M^{(n+1)}, M^{(n)}) \cong H_{n+1}(M^{(n+1)}, M^{(n)}) \cong C^{cw}_{n+1}(M)$ and similarly for N

so the cocycle $\gamma^{n+1}(f^* E) = f^* \gamma^{n+1}(E)$ $\#$

example:

recall a vector bundle $\begin{array}{c} \mathbb{R}^n \to E \\ \downarrow \\ M \end{array}$

has a k-frame $\Leftrightarrow$ structure group reduces to $GL(n-k)$

or first put a metric on E so it has str. group $O(n)$ then

$\left[\begin{array}{l} E \text{ has an orthonormal } k\text{-frame} \\ \Leftrightarrow \\ \text{structure group reduces to } O(n-k) \end{array}\right.$

in terms of principal bundles, let $\mathcal{F}(E)$ be the orthonormal frame bundle (i.e. the principal $O(n)$ bundle associated to E)

now $\left[\begin{array}{l} E \text{ has an orthonormal } k\text{-frame} \\ \Leftrightarrow \text{ lemma II.12} \\ \mathcal{F}(E)/O(n-k) \text{ has a section} \end{array}\right.$

the fibers of $\mathcal{F}(E)/O(n-k)$ are $O(n)/O(n-k) = V_{n,k}$

recall from Cor II.11

$$\pi_i(V_{n,k}) \cong \begin{cases} 0 & i < n-k \\ \mathbb{Z} & i = n-k \text{ even or } k=1 \\ \mathbb{Z}/2 & i = n-k \text{ odd} \end{cases}$$

unfortunately $\pi_1(M)$ does not necissarily act trivially on $\pi_{n-k}(V_{n,k})$ if it is $\mathbb{Z}$

but if we take the mod 2 reduction it will (any action on $\mathbb{Z}/2$ is the identity)

so we have a primary obstruction to a k-frame over the $n-k+1$ skeleton

$$\gamma_{n-k+1}(E) \in H^{n-k+1}(M; \pi_{n-k}(V_{n,k}) \bmod 2)$$

set $w_\ell(E) = \gamma_\ell(E) \in H^\ell(M; \mathbb{Z}/2)$

this is the ℓ^{th} Steifel-Whitney class of E

when ℓ even (so $n-k$ odd where $\ell = n-k+1$)

$w_\ell(E)$ is the primary obstruction to $\exists$ of an $(n-\ell+1)$ frame on $M^{(\ell-1)}$ that extends over $M^{(\ell)}$

in general it is a "reduction" of this

Fact: (Steenrod) the w_i determine all the primary obstructions

exercise:

Given
$$\begin{array}{ccc} \mathbb{R}^n & \to & E \\ & & \downarrow \\ & & M \end{array}$$

1) $w_1(E)=0 \iff \exists$ a n-frame over $M^{(0)}$ that extends over $M^{(1)}$

$\iff E$ orientable

2) if E orientable, then

$w_2(E)=0 \iff \exists$ an $(n-1)$-frame over $M^{(1)}$ that extends over $M^{(2)}$

$\iff \exists$ an n-frame over $M^{(1)}$ that extends over $M^{(2)}$

this is called a spin structure

example:

if $\begin{array}{c} \mathbb{R}^n \to E \\ \downarrow \\ M \end{array}$ is an oriented bundle

then $\pi_1(M)$ acts trivially on $\pi_{n-1}(V_{n,1}) \cong \mathbb{Z}$

exercise: Check this

so we get a primary obstruction

$e(E) \in H^n(M;\mathbb{Z})$

to the existence of a non-zero section

$e(E)$ is called the Euler class

exercise:

1) if $s: M \to E$ is any section and M a manifold then we can homotop s so it is transverse to the zero section $Z \subset E$ and

Poincaré Dual

$$e(E) = P.D.\left[s^{-1}(Z)\right]$$

2) $e(TM)([M]) = \chi(M)$

fundamental class of M

Euler characteristic

example:

let
$$\begin{array}{c} \mathbb{C}^n \to E \\ \downarrow \\ M \end{array}$$
be a vector bundle with structure group $GL(n;\mathbb{C})$

this is a "complex bundle"

(from above can assume str. gp. $U(n)$)

so the frame bundle $\mathcal{F}(E)$ can be taken to be a principal $U(n)$-bundle

as in the real case, $\mathcal{F}(E)$ will have a complex k-frame

$\Longleftrightarrow$

$\mathcal{F}(E)/_{U(n-k)}$ has a section

(this is a $U(n)/_{U(n-k)} = V_{n,k}(\mathbb{C})$ bundle)

from Cor II. 11

$$\pi_i(V_{n,k}(\mathbb{C})) \cong \begin{cases} 0 & i \leq 2(n-k) \\ \mathbb{Z} & i = 2(n-k)+1 \end{cases}$$

exercise: $\pi_1(M)$ acts trivially on $\pi_{2(n-k)+1}(V_{n,k}(\mathbb{C}))$ where we think of $V_{n,k}(\mathbb{C})$ as the fiber of $\mathcal{F}(E)/_{U(n-k)}$

thus the primary obstruction to a complex k-frame is

$$\gamma^{2(n-k)+2} \in H^{2(n-k)+2}(M; \underbrace{\pi_{2(n-k)+1}(V_{n,k}(\mathbb{C}))}_{\mathbb{Z}})$$

we define $c_k(E) = \gamma^{2k}(E) \in H^{2k}(M;\mathbb{Z})$
this is the k^{th} Chern class of E

clearly $c_k(E)$ is the obstruction to a complex $(n-k+1)$ frame on $M^{(2k-1)}$

that extends to $M^{(2k)}$

exercise: if E is a complex $\mathbb{C}^n$-bundle over M

1) $c_n(E) = e(E)$

2) $w_{2i+1}(E) = 0$ ($\Rightarrow$ complex bundles are oriented)

3) $w_{2i}(E) = c_i(E)$ mod 2

4) $c_1(E) = 0 \Leftrightarrow$ structure group of E reduces to $SU(n)$

"complex orientation"

5) if $\bar{E}$ is E with "conjugate complex structure" i.e. for $z \in \mathbb{C}$ multiply by $\bar{z}$

then $c_i(\bar{E}) = (-1)^i c(E)$

Hint: easy for $c_n(E)$, reduce to this see Milnor - Stasheff

there is one more "standard" characteristic class

given a real bundle $\mathbb{R}^n \to E$
$$\downarrow$$
$$M$$

then $E \otimes_{\mathbb{R}} \mathbb{C}$ is a complex vector bundle

the i^{th} Pontrjagin class of E is

$$p_i(E) = (-1)^i c_{2i}(E \otimes \mathbb{C}) \in H^{4i}(M;\mathbb{Z})$$

exercise:

1) Show $E \otimes \mathbb{C}$ and $\overline{E \otimes \mathbb{C}}$ are isomorphic use this to show $c_{2i+1}(E \otimes \mathbb{C})$ is 2-torsion

2) if E is an oriented $\mathbb{R}^{2n}$-bundle then $p_n(E) = e(E) \cup e(E)$

3) if E is a complex bundle and $E^{\mathbb{R}}$ denotes the underlying real bundle then $E^{\mathbb{R}} \otimes \mathbb{C} \cong E \otimes \bar{E}$

4) if E is a $\mathbb{C}^n$-bundle then

$$1 - p_1(E) + p_2(E) \dots \pm p_n(E) = (1 + c_1(E) + \dots + c_n(E)) \cup (1 - c_1(E) \dots \pm c_n(E))$$

e.g. $p_1(E) = c_1(E) \cup c_1(E) - 2c_2(E)$

Characteristic classes, in general, do not determine a bundle, but we do have

I) Complex line bundles are determined by c_1 <u>and</u> any $\alpha \in H^2(M)$ is c_1 of some $\mathbb{C}$-bundle

i.e. $\{\mathbb{C}\text{-bundles over } M\} \overset{1-1}{\Longleftrightarrow} H^2(M)$

II) $\mathbb{C}^2$-bundles are determined by c_1 and c_2 <u>and</u> $\forall\, (\alpha, \beta) \in H^2(M) \times H^4(M)$, $\exists$ a $\mathbb{C}^2$-bundle E s.t. $c_1(E) = \alpha$, $c_2(E) = \beta$

III) $SO(3)$-bundles are isomorphic $\Leftrightarrow$ w_2, p_1 agree

IV) $SO(4)$-bundles are isomorphic
$\Leftrightarrow$
w_2, p_1, e agree

exercise: prove the above

I) "easy" II) "easyish"

III), IV) harder

B. Characteristic classes

another way to think of Steifel-Whitney classes

Thm 5:

$\exists$ a unique function

$$w_i : Vect(M) \to H^i(M; \mathbb{Z}/2) \quad \forall M$$

satisfying

1) $w_i(f^*E) = f^* w_i(E) \quad \forall f : M \to N$
2) $w_0(E) = 1, \ w_i(E) = 0 \quad \forall i > \text{fiber dim } E$
3) $w(E_1 \oplus E_2) = w(E_1) \cup w(E_2)$

   where $w(E_i) = 1 + w_1(E_i) + w_2(E_i) + \dots$
4) $w_1(\gamma_n) \neq 0$ where γ_n is the universal line bundle over $\mathbb{R}P^n$

for 3) $E_1 \oplus E_2$ is called the direct sum of E_1 and E_2 and has fiber the direct sum of the fibers of E_1 and E_2